Simple Models in Supersymmetric Quantum Mechanics on a Graph

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Abstract

We study some sorts of dimensionally-deconstructed models for supersymmetric (Euclidean) quantum mechanics, or zero-dimensional field theory. In these models, we assign bosonic and fermionic variables to vertices and edges of a graph. We investigate a discrete version for the Gaussian model and the Wess-Zumino-type model on a graph. The topological index as a multiple integral is discussed on these models. In addition, we propose simple examples for supersymmetric extensions of the Lee-Wick model and the Galileon model. A model with two supersymmetries is also provided and generalization to 'local' supersymmtric models is examined.

1 Introduction

The non-perturbative effects play important roles in many aspects of quantum field theory. Particularly, they are considered to be crucial for breakdown of supersymmetry in field theories. An approach to understand the dynamical supersymmetry breaking is to study supersymmetric models in quantum mechanics, which was suggested by Witten [1]. In a certain sense, quantum mechanics defined through the path integral is equivalent to zero-dimensional field theory. Moreover, the view point of the path integral provides prospects for topological properties of dynamical models in many problems. On the other hand, supersymmetric field theory has topological nature of its own because of its cohomological structure [2]. Therefore, various models with supersymmetry in lower dimensions is worth studying due to mathematical interest.

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The present authors have examined field theories on a graph [3] and models with superfields on a graph [4], which are interpreted as extensions of dimensional deconstruction [5, 6]. The dimensional deconstruction is a powerful tool to analyze the higher-dimensional theory by adopting multiple fields in lower dimensions. Our models mentioned above consist of different kinds of fields on vertices and edges on a graph. Thus, we acquire an idea of 'deconstructing' one-dimensional theory by assigning the different 'multiplets' to vertices and edges. The details are shown in Section 4.

In the present paper, we propose various models for supersymmetric quantum mechanics on a graph. In particular, we provide analogue models for higher-derivative theories with supersymmetry.

Discrete models are also useful to make a functional integral well defined mathematically. The models on a graph have a continuum limit of discrete variables in simple restricted cases. Actually, similar cases are known by a lattice formulation of field theory in order to study non-perturbative effects through numerical simulations [7, 8, 9, 10]. Another related study can be found in recent literature on zero-dimensional models of matrix theory [11, 12, 13]. In the present paper, although we do not make further mention on connection to these approaches and models, we should keep our mind on possible development of our models by incorporating their technical methods. In general cases, our supersymmetric models on a graph do not have continuum limit. This feature is mathematically interesting and would be studied in future.

The organization of the present paper is as follows. In Section 2, we shall give a brief review of 'zero-dimensional' supersymmetry on a toy model. Section 3 is devoted to description of an algebraic aspect of graph theory, namely, introduction of matrices associated with a graph. After these preparation, we construct a simple model on a graph in Section 4. An analogue model for a self-interacting field theory is discussed in Section 5. In Section 6, we deal with analogue models for higher-derivative theory. In Section 7, we consider a sort of extension, which leads to a model with two supersymmetries. Based on this model, we explore the possibility of 'local' supersymmetry, requiring an individual parameter of transformation for each variables in Section 8. Finally, we give concluding remarks in Section 9.

2 Review of zero-dimensional supersymmetry

The simplest example [2] for supersymmetric quantum mechanics contains bosonic (commuting) variables ϕ, F and fermionic (anticommuting, Grassmann) variables ψ, $\bar{\psi}$.

The fermionic transformation is defined by using a supercharge Q as

$$Q\phi = \psi\,, \quad Q\psi = 0\,, \quad Q\bar{\psi} = F\,, \quad QF = 0\,. \tag{1}$$

It is easy to find nilpotency $Q^2 = 0$ in these relations. Then, the following

'action' S is invariant under the transformation induced by Q:

$$S[\phi,\psi,\bar{\psi},F] = F\cdot P(\phi) - \bar{\psi}P'(\phi)\psi - \frac{1}{2}F^2\,, \tag{2}$$

where P is a function of ϕ and P' is the first derivative of P. One can find that the action S is supersymmetric by construction, because S can be written in the form

$$S = Q\left[\bar{\psi}\left(P(\phi) - \frac{1}{2}F\right)\right]\,. \tag{3}$$

Here, we take F as an auxiliary variable, so the action can be read as

$$S[\phi,\psi,\bar{\psi}] = \frac{1}{2}P^2(\phi) - \bar{\psi}P'(\phi)\psi\,. \tag{4}$$

after elimination of the auxiliary variable F by its equation of motion. One can find this form of the action quite familiar, and also find that

$$Z = \int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}\int d\psi d\bar{\psi}\, e^{-S[\phi,\psi,\bar{\psi}]} \tag{5}$$

takes values ± 1 or 0 by case of the behavior of $P(\phi)$ in the limit of $\phi\to\pm\infty$. This is closely related to the Witten index [14].

In Section 4 and subsequent sections, we utilize a number of variables and construct some models in supersymmetric quantum mechanics on a graph; the method we adopt is very similar to the idea of dimensional deconstruction [5, 6].

3 Review of graph theory and matrices therein

In this section, we review some matrices which are very useful to describe models on a graph [3, 4].

Let $G(\mathcal{V},\mathcal{E})$ be a graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. An oriented edge $e=[v,v']$ connects two adjacent vertices $v=o(e)$ and $v'=t(e)$, where $o(e)$ is the origin of the edge e and $t(e)$ is the terminus of the edge e. The number of adjacent vertices of a vertex v is called the degree of v, and is expressed by d_v.

The incidence matrix E for a directed graph is defined by

$$E_{ve}(G) = \begin{cases} 1 & \text{if } v=o(e) \\ -1 & \text{if } v=t(e) \\ 0 & \text{otherwise} \end{cases}\,. \tag{6}$$

Similarly, the 'unoriented' incidence matrix B is defined by

$$B_{ve}(G) = \begin{cases} 1 & \text{if } v=o(e) \text{ or } v=t(e) \\ 0 & \text{otherwise} \end{cases}\,. \tag{7}$$

The graph Laplacian L is defined by

$$L_{vv'}(G) = \begin{cases} d_v & \text{if } v=v' \\ -1 & \text{if } [v,v']\in\mathcal{E} \\ 0 & \text{otherwise} \end{cases}\,. \tag{8}$$

There is an important relation between the incidence matrix and the graph Laplacian for a graph,

$$L_{vv'}(G) = (E(G)E(G)^T)_{vv'}\,. \tag{9}$$

Note that the Greek letter Δ is also used for representing the graph Laplacian in many textbooks. We will not use the symbol in this paper to avoid confusion with the difference operation.

For example, we consider a cycle graph. A cycle graph C_N is a set of N ($N \geq 3$) vertices lined up along a circle with edges between each vertex and its adjacent ones on each side. The incidence matrix for a cycle graph C_N is given by

$$E(C_N) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \tag{10}$$

The cycle graph considered here is a closed circuit along with one direction of edges, i.e., any vertex is an origin of one edge and a terminus of another edge at the same time. The transposed matrix of the incidence matrix can play a role of a difference operator.

The unoriented incidence matrix for a cycle graph C_N is

$$B(C_N) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}. \tag{11}$$

The graph Laplacian for C_N is given by

$$L(C_N) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{12}$$

One can find a similarity that $\sum_{v'} L_{vv'} f_{v'} \approx -\partial_t^2 f(t)$, where the parameter t is discretized and is expressed by a single sequence of v. It is easy to compute the eigenvalue of $L(C_N)$, which is found to be

$$4\sin^2\frac{\pi p}{N} \qquad (p = 0, 1, 2, \ldots, N-1)\,. \tag{13}$$

We can therefore see that the continuum limit ($N \to \infty$ and $\ell = Na$ =constant) leads to $(2\pi/\ell)^2$ as the eigenvalue of the $a^{-2}L(C_N)$, where a is a distance scale or 'lattice spacing', namely t is regarded as na $(0 \leq n < N)$.

Another well-known graph is the path graph P_N, which possesses N vertices and $N-1$ edges. The path graph has two ends (where $d_v = 1$) but all the other $N-2$ vertices have degree two ($d_v = 2$). The incidence matrix for a path graph with a definite direction is given by

$$E(P_N) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} . \tag{14}$$

The unoriented incidence matrix for a path graph P_N is similarly given as an $N \times (N-1)$ matrix, whereas the graph Laplacian for P_N is given by the following $N \times N$ matrix:

$$L(C_N) = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} . \tag{15}$$

(The graph Laplacian sometimes appears in textbooks of elementary dynamics, to explain the vibration of a ball-spring system!)

4 Construction of supersymmetric quantum mechanics on a graph: Gaussian models

Suppose $G(\mathcal{V}, \mathcal{E})$ be a simple graph. We assign a scalar variable ϕ_v and a fermionic variable ψ_v to each vertex v of the graph, whereas a fermionic variable $\bar{\psi}_e$ and a bosonic variable F_e to each edge e of the graph. Consider the following 'action':

$$S[\phi, \psi, \bar{\psi}, F] = \sum_{e \in \mathcal{E}} \left[F_e P_e(\{\phi_v\}) - \bar{\psi}_e \sum_{v \in \mathcal{V}} \frac{\partial P_e(\{\phi_v\})}{\partial \phi_v} \psi_v - \frac{1}{2} F_e F_e \right] , \tag{16}$$

where P_e is functions of ϕ_v.

The action S is invariant under the supersymmetry transformation

$$Q\phi_v = \psi_v \,, \quad Q\psi_v = 0 \,, \quad Q\bar{\psi}_e = F_e \,, \quad QF_e = 0 \,. \tag{17}$$

Note that $Q^2 = 0$ is ensured. Note also that the supersymmetry transformation does not include difference operators (matrices) in the present construction.

Because the action is given by

$$S = Q\left[\sum_{e\in\mathcal{E}} \bar{\psi}_e \left(P_e(\{\phi_v\}) - \frac{1}{2}F_e\right)\right] , \tag{18}$$

the invariance under the fermionic transformation is trivial.

To define a Gaussian (free) model, we specify P_e by a linear combination of a few ϕ's:

$$P_e(\{\phi_v\}) = \sum_{v\in\mathcal{V}} P_{ev}\phi_v , \tag{19}$$

where P_{ev} is a constant matrix. The elimination of the auxiliary variables yields the action for a free scalar and fermions:

$$S[\phi,\psi,\bar{\psi}] = \sum_{v,v'\in\mathcal{V}}\sum_{e\in\mathcal{E}} \frac{1}{2}\phi_v P^T_{ve}P_{ev'}\phi_{v'} - \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}} \bar{\psi}_e P_{ev}\psi_v , \tag{20}$$

where ϕ is a real scalar, ψ and $\bar{\psi}$ are fermions.

Here we consider the matrix as a combination of incidence matrices of a graph:

$$P_{ev} = E^T + mB^T , \tag{21}$$

where E and B are the oriented and unoriented incidence matrices for a graph. This is equivalent to the following expression:

$$P_e = \sum_{v\in\mathcal{V}} P_{ev}\phi_v = (1+m)\phi_{o(e)} - (1-m)\phi_{t(e)} . \tag{22}$$

If we consider a cycle graph C_N, the eigenvalues of matrices can be easily obtained and turns out to be[1]

$$\det(E^T(C_N) + mB^T(C_N)) = \prod_{p=0}^{N-1}[(1+m) - (1-m)e^{i\frac{2\pi p}{N}}] , \tag{23}$$

and

$$\begin{aligned}&\det[(E(C_N)+mB(C_N))(E^T(C_N)+mB^T(C_N))]\\ &= \prod_{p=0}^{N-1}\left[4(1-m^2)\sin^2\frac{\pi p}{N} + 4m^2\right] .\end{aligned} \tag{24}$$

Thus, in this case, the 'partition function'

$$Z = \int [D\phi][D\psi][D\bar{\psi}]\, e^{-S[\phi,\psi,\bar{\psi}]} , \tag{25}$$

[1] Similar procedure can be carried out for a graph $G = C_{N_1} \cup C_{N_2} \cup \cdots$.

where $[D\phi] = \prod_{v\in\mathcal{V}} \frac{d\phi_v}{\sqrt{2\pi}}$, $[D\psi] = \prod_{v\in\mathcal{V}} d\psi_v$ and $[D\psi] = \prod_{e\in\mathcal{E}} d\bar{\psi}_e$, can be explicitly evaluated as a Gaussian multiple integral and becomes

$$Z = \frac{\det(E^T(C_N) + mB^T(C_N))}{\sqrt{\det[(E(C_N) + mB(C_N))(E^T(C_N) + mB^T(C_N))]}} = 1\,. \tag{26}$$

This answer is in agreement with that of Ref. [10].

For a cycle graph, the transpose of the incidence matrix E^T corresponds to a difference operator, which maps the difference between variables assigned on adjacent vertices to the edge connecting the vertices. Of course, the 'lattice spacing', which is needed for correspondence to continuum theory, is considered as being omitted here and can be recovered by rescaling variables appropriately.

One can find that the transpose of the unoriented incidence matrix provides the 'mass term' or mass matrix in the action. It is worth noting that the mass term is slightly 'non-local', due to its concerning with the variables on the nearest neighbor vertices (or edges). This is however a natural choice because of the assignment of variables to make the supersymmetry apparent.

5 Superpotentials on a cycle graph

Next, we will examine non-linear models. For simplicity, in this section, we mainly consider a cycle graph C_N and generalization to other graphs will be briefly discussed later. Thus, we simply denote the incidence matrix as E.

Consider the case that P_e in (16) is a sum of the 'difference' $\sum_{v\in\mathcal{V}} E^T_{ev}\phi_v$ and non-linear functions $\varDelta W_e(\{\phi_v\})$. In Ref. [15], we considered similar discrete models in which a kink-shaped configuration exists. This is equivalent to choosing the function P_e in the present paper as

$$\begin{aligned} P_e &= \sum_{v\in\mathcal{V}} E^T_{ev}\phi_v + \varDelta W_e(\{\phi_v\}) \\ &= \phi_{o(e)} - \phi_{t(e)} + g\left(1 - \frac{\phi^2_{o(e)} + \phi_{o(e)}\phi_{t(e)} + \phi^2_{t(e)}}{3}\right), \end{aligned} \tag{27}$$

up to coefficients. Here g is a coupling constant.

The advantage of this choice is the fact that the bosonic action reduces to the simple sum of the 'kinetic' term and the 'potential' term. That is,

$$\sum_{e\in\mathcal{E}} (P_e)^2 = \sum_{e\in\mathcal{E}} (\phi_{o(e)} - \phi_{t(e)})^2 + g^2 \sum_{e\in\mathcal{E}} \left(1 - \frac{\phi^2_{o(e)} + \phi_{o(e)}\phi_{t(e)} + \phi^2_{t(e)}}{3}\right)^2, \tag{28}$$

because

$$\begin{aligned} &\sum_{e\in\mathcal{E}} (\phi_{o(e)} - \phi_{t(e)})\left(1 - \frac{\phi^2_{o(e)} + \phi_{o(e)}\phi_{t(e)} + \phi^2_{t(e)}}{3}\right) \\ &= \sum_{e\in\mathcal{E}} \left[\left(\phi_{o(e)} - \frac{\phi^3_{o(e)}}{3}\right) - \left(\phi_{t(e)} - \frac{\phi^3_{t(e)}}{3}\right)\right] = 0\,. \end{aligned} \tag{29}$$

For the cycle graph, in which every vertex has degree two, the cancellation of the cross term is obvious. Actually, the cancellation is attained if $\mathit{\Delta} W_e(\phi_{o(e)}, \phi_{t(e)}) = \mathit{\Delta} W_e(\phi_{t(e)}, \phi_{o(e)})$ for a cycle graph, owing to its homogeneous structure. In the above example, however, cancellation occurs only in the terms corresponding to the nearest-neighbor edges. We can imagine such a 'locality' in a model if $\mathit{\Delta} W_e$ is expressed by

$$\mathit{\Delta} W_e \equiv \frac{W(\phi_{o(e)}) - W(\phi_{t(e)})}{\phi_{o(e)} - \phi_{t(e)}} \,. \tag{30}$$

Note that for $W(\phi) = m\phi^2$, we get $\mathit{\Delta} W_e = m \sum_{v \in \mathcal{V}} B^T_{ev} \phi_v$, which has appeared in the Gaussian model in the previous section.

Now, we examine the topological index, the partition function of the model. First, we consider a parametrized partition function

$$Z(t) = \int [D\phi][D\psi][D\bar{\psi}][DF] e^{-S[\phi,\psi,\bar{\psi},F]+tQV} \,, \tag{31}$$

which turns out to be independent of the parameter t, so $Z(t) = Z$, if $QS = 0$ and $Q^2 = 0$. In the expression of $Z(t)$, F_e should be replaced as $F_e \to iF_e$ for convergence of integration, so the action becomes

$$S[\phi, \psi, \bar{\psi}, F] = \sum_{e \in \mathcal{E}} \left[iF_e P_e(\{\phi_v\}) - \bar{\psi}_e \sum_{v \in \mathcal{V}} \frac{\partial P_e(\{\phi_v\})}{\partial \phi_v} \psi_v + \frac{1}{2} F_e F_e \right] . \tag{32}$$

Now, we take $tQV = \frac{1}{2} \sum_{e \in \mathcal{E}} F_e^2$. After carrying out fermionic integrations, we obtain

$$\begin{aligned} Z &= \int [D\phi][DF] \left| \frac{\partial P_e(\{\phi_v\})}{\partial \phi_v} \right| \exp\left[-i \sum_{e \in \mathcal{E}} F_e P_e \right] \\ &= \int_{\Omega_P} [DP] \, \delta^N(P) \,. \end{aligned} \tag{33}$$

Here we denote $\int_{\Omega_P} [DP] = \int [D\phi] \left| \frac{\partial P_e(\{\phi_v\})}{\partial \phi_v} \right|$, where the determinant is the Jacobian. Note that the integration region Ω_P does not need to be R^N. Therefore, Z gives the winding number of map $P_e(\{\phi_v\})$.[2] We find that, under the assumption of 'locality', for $W(\phi) \propto \phi^n$ (n is an integer), $Z = 0$ if n is odd while $Z = 1$ (or -1) if n is even. This is the same as the result on usual supersymmetric quantum mechanics. It is noteworthy that the analysis of the partition function is easy if $\mathit{\Delta} W_e$ is a function only of $\phi_{o(e)}$ and $\phi_{o(e)}$.

Before closing this section, we give a comment on generalization to the model for quantum mechanics on a general graph. The separation of the kinetic term and the potential term is possible if we elaborate to cancel the cross term in $\sum_{e \in \mathcal{E}} P_e^2$. To accomplish the 'local' cancellation as in the case with cycle graphs,

[2] The manifold represented by the potential in the continuum limit is usually taken to be connected, so the winding number ($R^N \to \Omega_P$, where $R^N \cup \{\infty\} \approx \Omega_P \cup \{\infty\} \approx S^N$) should be $0, \pm 1$.

we should choose an Euler graph. For every vertex v in an Euler graph, the number of edges satisfying $o(e) = v$ equals to the number of edges satisfying $t(e) = v$. The cancellation thus occurs at every vertex.

6 'Higher-derivative' models

Recently, higher-derivative models in field theory are eagerly studied in particle physics [16] and cosmology [17, 18, 19]. Their supersymmetric generalizations are also investigated by many authors [20, 21, 22, 23, 24]. In this section, we consider quantum-mechanical analogue models of supersymmetric higher-derivative theories.

Consider $K_{ee'}(\{\phi_v\})$, a matrix of functions on ϕ_v and assume that $K_{ee'}$ is a symmetric matrix, i.e., $K_{ee'} = K_{e'e}$.

Now, the action for the supersymmetric model including $K_{ee'}$ becomes

$$\begin{aligned} S &= Q\left[\sum_{e,e'\in\mathcal{E}} \bar{\psi}_e K_{ee'}(\{\phi_v\})\left(P_{e'}(\{\phi_v\}) - \frac{1}{2}F_{e'}\right)\right] \\ &= \sum_{e,e'\in\mathcal{E}}\left[F_e K_{ee'} P_{e'} - \bar{\psi}_e \sum_{v\in\mathcal{V}} \frac{\partial(K_{ee'}P_{e'})}{\partial\phi_v}\psi_v + \frac{1}{2}\bar{\psi}_e \sum_{v\in\mathcal{V}} \frac{\partial K_{ee'}}{\partial\phi_v}\psi_v F_{e'} \right. \\ &\qquad \left. -\frac{1}{2}F_e K_{ee'} F_{e'}\right]. \end{aligned} \tag{34}$$

The equation of motion for F_e turns out to be

$$\sum_{e'\in\mathcal{E}} K_{ee'}P_{e'} + \sum_{e'\in\mathcal{E}}\sum_{v\in\mathcal{V}} \frac{1}{2}\bar{\psi}_{e'} \frac{\partial K_{e'e}}{\partial\phi_v}\psi_v - \sum_{e'\in\mathcal{E}} K_{ee'}F_{e'} = 0\,, \tag{35}$$

and reduces to

$$F_e = P_e + \sum_{e',e''\in\mathcal{E}}\sum_{v\in\mathcal{V}} \frac{1}{2} K^{-1}_{ee''} \frac{\partial K_{e''e'}}{\partial\phi_v} \bar{\psi}_{e'}\psi_v\,. \tag{36}$$

Substitution of the equation simplifies the action to

$$\begin{aligned} S &= \sum_{e,e'\in\mathcal{E}}\left[\frac{1}{2}P_e K_{ee'} P_{e'} - \bar{\psi}_e \sum_{v\in\mathcal{V}} K_{ee'} \frac{\partial P_{e'}}{\partial\phi_v}\psi_v \right. \\ &\qquad \left. + \frac{1}{8}\sum_{e'',e'''\in\mathcal{E}}\sum_{v,v'\in\mathcal{V}} \bar{\psi}_e\psi_v \frac{\partial K_{ee''}}{\partial\phi_v} K^{-1}_{e''e'''} \frac{\partial K_{e'''e'}}{\partial\phi_{v'}} \bar{\psi}_{e'}\psi_{v'}\right]. \end{aligned} \tag{37}$$

If we incorporate the incidence matrix E and its transpose E^T as the 'difference operators' into $K_{ee'}$, we can construct discrete analogue model for higher-derivative theories.

The partition function can be evaluated as in a similar manner shown in the previous section. In the present case, we choose $tQV = \frac{1}{2}\sum_{e\in\mathcal{E}} F_e(KP)_e$. (Here and hereafter the sum over edges (or vertices) is not indicated if the multiplication of matrices is obvious.) Thus, we get

$$\begin{aligned} Z &= \int [D\phi][DF] \left| \frac{\partial (KP)_e}{\partial \phi_v} \right| \exp\left[-i \sum_{e\in\mathcal{E}} F_e (KP)_e \right] \\ &= \int_{\Omega_{KP}} [D(KP)]\, \delta^N(KP)\,, \end{aligned} \tag{38}$$

where the inner sum over edges is suppressed.

6.1 Lee-Wick model

More than forty years ago, Lee and Wick and the other authors considered higher-derivative action in order to avoid the infinity in quantum field theory [25, 26, 27, 28]. Recently, the idea has been revived for solving the hierarchy problem [16]. In this subsection, we provide a discrete model for quantum mechanics with 'higher derivatives'.

We adopt the following matrix as $K_{ee'}$:

$$K_{ee'} = \delta_{ee'} + \alpha (E^T E)_{ee'}\,, \tag{39}$$

where $\delta_{ee'}$ denotes the identity matrix and α is a constant. Moreover, we consider the simplest case, therefore we take

$$P_{e'} = \sum_{v\in\mathcal{V}} E^T_{e'v}\phi_v\,. \tag{40}$$

The supersymmetric action then becomes

$$\begin{aligned} S[\phi,\psi,\bar\psi,F] &= \sum_{e\in\mathcal{E},v\in\mathcal{V}} F_e (E^T + \alpha E^T E E^T)_{ev}\phi_v \\ &\quad - \sum_{e\in\mathcal{E},v\in\mathcal{V}} \bar\psi_e (E^T + \alpha E^T E E^T)_{ev}\psi_v \\ &\quad - \sum_{e,e'\in\mathcal{E}} \frac{1}{2} F_e (\delta_{ee'} + \alpha E^T E)_{ee'} F_{e'}\,. \end{aligned} \tag{41}$$

Eliminating the auxiliary fields F_e, we are left with

$$\begin{aligned} S[\phi,\psi,\bar\psi] &= \frac{1}{2}\sum_{v',v\in\mathcal{V}} \phi_{v'} (EE^T + \alpha E E^T E E^T)_{v'v}\phi_v \\ &\quad - \sum_{e\in\mathcal{E},v\in\mathcal{V}} \bar\psi_e (E^T + \alpha E^T E E^T)_{ev}\psi_v\,. \end{aligned} \tag{42}$$

This model is an analogue of Lee-Wick theory, whose Lagrangian is $\mathcal{L} = -\frac{1}{2}\phi\nabla^2(1+M^{-2}\nabla^2)\phi - \bar{\psi}D(1+M^{-2}\nabla^2)\psi$, where ∇^2 is the Laplacian and D denotes the Dirac operator [20, 21, 22, 23], with $M^{-2} \sim \alpha a^2$ (a is a length scale).

The topological value for the partition function of this model turns out to be unity for $\varDelta W_e \neq 0$, in general. The variable $(KP)_e$ in (38) depends not only on $\varDelta W_e$ but also on $\varDelta W_{\tilde{e}}$, where $\tilde{e}$ is the edge whose end is the same with one of the edge e. Thus, even if some $\varDelta W_e$ takes a restricted value bound above or below, the value for $(KP)_e$ can run over from $-\infty$ to ∞, in general.

6.2 Galileon model

In cosmology, scalar field theories with higher-derivative terms are studied with much interest. The DGP-like Galileon term, which is cubic in a scalar field with four derivative operators, is motivated from D-brane theory [17]. The generalized Galileon field theory has been developed recently [18, 19]. Furthermore, an attempt to supersymmetrize the Galileon models appears in Ref. [24].

Here we choose $K_{ee'}$ for an analogue model:

$$K_{ee'} = \delta_{ee'} + \beta_1 \sum_{v\in\mathcal{V}} E^T_{ev}\phi_v E_{ve'} + \frac{\beta_2}{4}[(E^TE)_{ee'}(B\phi)_{e'} + (B\phi)_e(E^TE)_{ee'}]\,, \tag{43}$$

where the coefficients β_1 and β_2 are constant. Considering now the simplest case:

$$P_{e'} = \sum_{v\in\mathcal{V}} E^T_{e'v}\phi_v\,, \tag{44}$$

with help of the following identity, which is trivial if one rewrite this using $v = o(e)$ and $v = t(e)$:

$$2\sum_{v\in\mathcal{V}} E^T_{ev} f_v g_v = \sum_{v\in\mathcal{V}} E^T_{ev} f_v \sum_{v'\in\mathcal{V}} B^T_{ev'} g_{v'} + \sum_{v\in\mathcal{V}} E^T_{ev} g_v \sum_{v'\in\mathcal{V}} B^T_{ev'} f_{v'}\,, \tag{45}$$

we obtain the part of the action for scalars:

$$\begin{aligned} S_B[\phi] &= \frac{1}{2}\sum_{e,e'\in\mathcal{E}} P_e K_{ee'} P_{e'} \\ &= \frac{1}{2}\sum_{v,v'\in\mathcal{V}} \phi_v(EE^T)_{vv'}\phi_{v'} + \frac{\beta_1}{2}\sum_{v,v',v''\in\mathcal{V}} \phi_v(EE^T)_{vv'}\phi_{v'}(EE^T)_{v'v''}\phi_{v''} \\ &\quad + \frac{\beta_2}{4}\sum_{v,v',v''\in\mathcal{V}} \phi_v(EE^TEE^T)_{vv'}\phi_{v'}\phi_{v'}\,. \end{aligned} \tag{46}$$

The bilinear operator

$$\Gamma(f,g)_v = \frac{1}{2}\sum_{v'\in\mathcal{V}} \{L_{vv'}f_{v'}g_{v'} - f_v L_{vv'}g_{v'} - g_v L_{vv'}f_{v'}\} \tag{47}$$

has been introduced by Chung, Lin and Yau recently [29, 30] (but they used the other type of Laplacian as L). The correspondence to continuum theory is

known as $\Gamma(f,g)_v \sim -a^2\partial f\cdot\partial g$ (where a is a lattice spacing). Therefore, if we choose $\beta_2=-\beta_1=\beta$, the bosonic part of the action can be read as

$$S_B[\phi] \;=\; \frac{1}{2}\sum_{v\in\mathcal{V}}(E^T\phi)_v(E^T\phi)_v+\frac{1}{2}\beta\sum_{v\in\mathcal{V}}(EE^T\phi)_v\Gamma(\phi,\phi)_v\,, \tag{48}$$

and the action for continuum theory derived from this can be written as $S\sim\int dt[\frac{1}{2}(\partial\phi)^2+\frac{\beta a^2}{2}\partial^2\phi(\partial\phi)^2]$, which is the action for DGP-type Galileon.

If we regard the incidence matrice as a difference operator, a continuum limit can be achieved up to some distance scale a and we get

$$K_{ee'}\Rightarrow K=1+\beta_1 a^2\overleftarrow{\partial}\,\phi\,\overrightarrow{\partial}+\frac{\beta_2 a^2}{2}[\overleftarrow{\partial}\overrightarrow{\partial}\,\phi+\phi\,\overleftarrow{\partial}\overrightarrow{\partial}]\,, \tag{49}$$

where the arrows ($\rightarrow$, $\leftarrow$) indicate the direction which the derivative operator acts on. It will be interesting to examine the system governed by the action $S=\int dt\frac{1}{2}P(\phi)K(\phi)P(\phi)$ with $P(\phi)=\overrightarrow{\partial}\,\phi+W'(\phi)$, where $W'(\phi)$ is a certain function of ϕ. We also imagine generalization to higher-dimensional scalar models. In such a manner, the quantum mechanical model with rigid supersymmetry provides a new insight to model building in field theory. In any case, because investigation of continuum models is beyond the scope of the present paper, these subjects are left for future study.

7 Models with two supersymmetries

The models so far considered is constructed by variables assigned to vertices and edges of a graph. In general graphs, the numbers of vertices and edges are different, whereas they coincides with each other for cycle graphs. Therefore, there are 'zero modes' of the matrices associated with general graphs a priori. As a zero-dimensional model, the partition function becomes trivial in such a case without discarding zero-mode contributions.

In this section, we improve the assignment of the bosonic and fermionic variables; both variables are assigned vertices as well as edges. This extension enables us to consider two supersymmetries in a model. This formulation is useful to attempt to consider 'local supersymmetry' in the next section, but we will find difficulty in the establishment.

Suppose that scalar variables are assigned to each edge as well as to each vertex, that is

$$\phi=\begin{pmatrix}\phi_v\\ \phi_e\end{pmatrix}, \tag{50}$$

and similarly ψ, $\bar\psi$ and F are put on both vertices and edges of a graph:

$$\psi=\begin{pmatrix}\psi_v\\ \psi_e\end{pmatrix},\quad \bar\psi=\begin{pmatrix}\bar\psi_v\\ \bar\psi_e\end{pmatrix},\quad F=\begin{pmatrix}F_v\\ F_e\end{pmatrix}. \tag{51}$$

Now we introduce two fermionic transformation induced by supercharges Q_1 and Q_2. They are expressed as

$$Q_1\phi = \psi\,, \quad Q_1\psi = 0\,, \quad Q_1\bar{\psi} = F\,, \quad Q_1 F = 0\,, \tag{52}$$

and

$$Q_2\phi = \bar{\psi}\,, \quad Q_2\psi = -F\,, \quad Q_2\bar{\psi} = 0\,, \quad Q_2 F = 0\,. \tag{53}$$

One can see that the supersymmetry algebra takes the form

$$Q_1^2 = Q_2^2 = 0\,, \qquad Q_1 Q_2 + Q_2 Q_1 = 0\,. \tag{54}$$

To construct an invariant 'action' of the variables, we define the following matrix for convenience:

$$\mathbf{E} = \begin{pmatrix} O & -E \\ E^T & O \end{pmatrix}\,. \tag{55}$$

Note that this square matrix satisfies $\mathbf{E}^T = -\mathbf{E}$. The following action is invariant under the two transformations:

$$\begin{aligned} S[\phi,\psi,\bar{\psi},F] &= Q_1\left[\bar{\psi}^T\left(\mathbf{E}\phi - \frac{1}{2}F\right)\right] = Q_2\left[\psi^T\left(\mathbf{E}\phi + \frac{1}{2}F\right)\right] \\ &= F^T\mathbf{E}\phi - \bar{\psi}^T\mathbf{E}\psi - \frac{1}{2}F^T F\,. \end{aligned} \tag{56}$$

Eliminating the auxiliary fields F, we obtain

$$S[\phi,\psi,\bar{\psi}] = \frac{1}{2}(\mathbf{E}\phi)^T\mathbf{E}\phi - \bar{\psi}^T\mathbf{E}\psi\,. \tag{57}$$

This action has a very similar form to the action for free fields.

Conversely, we can suppose a simpler system by identifying the two fermionic species as $\psi = \bar{\psi} = \Psi$. Then the action

$$S_0[\phi,\Psi] = \frac{1}{2}(\mathbf{E}\phi)^T\mathbf{E}\phi - \frac{1}{2}\Psi^T\mathbf{E}\Psi\,. \tag{58}$$

is invariant under a fermionic transformation induced by a supercharge Q, i.e., $QS_0[\phi,\Psi] = 0$, provided that the transformation rules are given by

$$Q\phi = \Psi\,, \qquad Q\Psi = (\mathbf{E}\phi)\,. \tag{59}$$

Note that $Q^2 = \mathbf{E}$.

The simple action (58) is useful to investigate the possibility of introducing further coupling to other variables. We will examine such a case via considering 'local' supersymmetry in the next section.

8 A 'locally' supersymmetric model?

A simple locally supersymmetric model is offered by van Nieuwenhuizen [31, 32, 33]. His model contains a massless free scalar field and a massless free fermionic field. We now come to an idea of constructing a 'locally' supersymmetric model which has the same structure as a simple model, by extending the last model. Therefore, we consider the last model and the 'local' definition of supersymmetry.

8.1 difficulty in 'local' models

We first restrict ourselves on a model on a cycle graph C_N here. We start with defining the 'local' super-transformation:

$$\delta\phi_c = \epsilon_c \Psi_c\,, \quad \delta\Psi_c = (\mathbf{E}\phi)_c(\tilde{\epsilon})_c\,, \tag{60}$$

where $c = 1, \ldots, 2N$, $\tilde{\epsilon} \equiv \frac{1}{2}\mathbf{B}\epsilon$, with

$$\mathbf{B} = \begin{pmatrix} O & B(C_N) \\ B(C_N)^T & O \end{pmatrix}. \tag{61}$$

The 'locality' we consider here should be the property that the transformation on a variable includes a limited number of variables in the neighbor vertices and edges.

Then, the variation of the action (58) becomes

$$\begin{aligned}\delta S_0 &= (\mathbf{E}\phi)^T\mathbf{E}\delta\phi - \Psi^T\mathbf{E}\delta\Psi \\ &= \sum_c \left[(\mathbf{E}\phi)_c \sum_a \mathbf{E}_{ca}\epsilon_a\Psi_a + (\mathbf{E}\Psi)_c(\mathbf{E}\phi)_c(\tilde{\epsilon})_c\right] \\ &= \sum_c \left[(\mathbf{E}\epsilon)_c(\mathbf{E}\phi)_c(\tilde{\Psi})_c\right], \end{aligned} \tag{62}$$

where $\tilde{\Psi} \equiv \frac{1}{2}\mathbf{B}\Psi$. Up to now, the definition of the 'local' transformation (60) seems to be good, nevertheless the transformation is slightly non-local because of the use of $\tilde{\epsilon}$. The identity (45) has been used in the second line of Eq. (62) and it enables us to write down the result as a single term.

To compensate this variation, we introduced a new Grassmann variable λ_c, which has its own variation

$$\delta\lambda_c = (\mathbf{E}\epsilon)_c + \cdots\,, \tag{63}$$

and we add a term into the action:

$$S_N = -\sum_c \left[\lambda_c(\mathbf{E}\phi)_c(\tilde{\Psi})_c\right]. \tag{64}$$

This is the very orthodox way to obtain local symmetries in field theory. The actual variation of the additional term can be found as:

$$\begin{aligned}\delta S_N &= -\sum_c \left[(\mathbf{E}\epsilon)_c(\mathbf{E}\phi)_c(\tilde{\Psi})_c\right] \\ &- \sum_c \left[\lambda_c \sum_a \mathbf{E}_{ca}\epsilon_a\Psi_a(\tilde{\Psi})_c\right] \\ &- \sum_c \left[\lambda_c(\mathbf{E}\phi)_c \sum_a \frac{1}{2}\mathbf{B}_{ca}(\mathbf{E}\phi)_a(\tilde{\epsilon})_a\right]. \end{aligned} \tag{65}$$

The first term of course cancels δS_0 given above. The second term in Eq. (65) can be removed if we consider the additional variation in $\delta\Psi_c$ as

$$\delta\Psi_c = (\mathbf{E}\phi)_c(\tilde{\epsilon})_c - \lambda_c(\widetilde{\Psi\epsilon})_c\,. \tag{66}$$

To see this, we use the identity (45) and $\lambda_c\lambda_c = \Psi_c\Psi_c = 0$.

If we attempt to cancel the third term, we need a new variable h_{ab}, which has the variation

$$\delta h_{ab} = \frac{1}{4}\mathbf{B}_{ab}(\tilde{\epsilon}_a\lambda_b + \tilde{\epsilon}_b\lambda_a)\,, \tag{67}$$

and the additional action

$$S_s = -\sum_{a,b}\left[h_{ab}(\mathbf{E}\phi)_a(\mathbf{E}\phi)_b\right]\,. \tag{68}$$

The actual variation of S_s becomes

$$\begin{aligned}
\delta S_s &= \sum_c\left[\lambda_c(\mathbf{E}\phi)_c\sum_a\frac{1}{2}\mathbf{B}_{ca}(\mathbf{E}\phi)_a(\tilde{\epsilon})_a\right] \\
&- 2\sum_{a,b}\left[h_{ab}(\mathbf{E}\phi)_a\sum_c\mathbf{E}_{bc}\epsilon_c\Psi_c\right] \\
&= \sum_c\left[\lambda_c(\mathbf{E}\phi)_c\sum_a\frac{1}{2}\mathbf{B}_{ca}(\mathbf{E}\phi)_a(\tilde{\epsilon})_a\right] \\
&- 2\sum_{a,b}\left[h_{ab}(\mathbf{E}\phi)_a\{(\mathbf{E}\epsilon)_b\tilde{\Psi}_b + \tilde{\epsilon}_b(\mathbf{E}\Psi)_b\}\right]\,.
\end{aligned} \tag{69}$$

In the model of van Nieuwenhuizen [31, 32, 33], the field h is of course locally coupled to the other fields and so the corresponding term of the last term in Eq. (69) can be canceled by the modification of the variation $\delta\lambda_c$. In our case, however, the term has the contribution of the neighborhood and next-to-neighbor variables, so the cancellation needs another new term.

We have already seen the 'non-locality' at the introduction of h_{ab}; thus, it is found to be difficult to obtain the symmetric model.

The origin of the difficulty is similarly in the fact that the Leibnitz rule does not hold for difference operators [34] in lattice field theory. Therefore, the 'non-locality' grows as the compensating procedure shown above is advanced.

8.2 a concise model on a smallest graph

The approach to finding a 'local' supersymmetric model in the previous subsection has failed for models on a general cycle graph. For a finite graph, however, the procedure of 'supersymmetrization' closes in finite steps; unfortunately, the continuous limit has no sense of course in this case. In the present section, we

demonstrate the construction of the model on a smallest path graph, P_2. In this case, the matrix $\mathbf{E}$ becomes

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{70}$$

which acts on $\phi = (\phi_1 \ \phi_2 \ \phi_3)^T$ and $\Psi = (\Psi_1 \ \Psi_2 \ \Psi_3)^T$. Therefore, the starting action (58) is found to be

$$\begin{aligned} S_0 &= \frac{1}{2}(\mathbf{E}\phi)^T \mathbf{E}\phi - \frac{1}{2}\Psi^T \mathbf{E}\Psi \\ &= \frac{1}{2}(\phi_1 - \phi_2)^2 + \phi_3^2 - \Psi_3(\Psi_1 - \Psi_2) \,. \end{aligned} \tag{71}$$

At the first time, the super-transformation is defined as follows:

$$\delta\phi_1 = \epsilon_1\Psi_1 \,, \quad \delta\phi_2 = \epsilon_2\Psi_2 \,, \quad \delta\phi_3 = \epsilon_3\Psi_3 \,, \tag{72}$$

$$\delta\Psi_1 = -\phi_3\epsilon_3 \,, \quad \delta\Psi_2 = +\phi_3\epsilon_3 \,, \quad \delta\Psi_3 = (\phi_1 - \phi_2)\frac{1}{2}(\epsilon_1 + \epsilon_2) \,, \tag{73}$$

which coincides with the 'global' transformation (59) if $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$.

Then, the variation δS_0 turns out to be

$$\delta S_0 = (\epsilon_1 - \epsilon_2)(\phi_1 - \phi_2)\frac{1}{2}(\Psi_1 + \Psi_2). \tag{74}$$

This term is expected to be canceled by the variation of the additional action S_N, where

$$S_N = -\lambda(\phi_1 - \phi_2)\frac{1}{2}(\Psi_1 + \Psi_2) \tag{75}$$

with

$$\delta\lambda = \epsilon_1 - \epsilon_2 \,. \tag{76}$$

Now, the variation of the action $S_0 + S_N$ becomes

$$\begin{aligned} \delta(S_0 + S_N) &= -\lambda(\epsilon_1\Psi_1 - \epsilon_2\Psi_2)\frac{1}{2}(\Psi_1 + \Psi_2) - \lambda(\phi_1 - \phi_2)\frac{1}{2}(-\phi_3\epsilon_3 + \phi\epsilon_3) \\ &= -\lambda\frac{1}{2}(\epsilon_1 + \epsilon_2)\Psi_1\Psi_2 \,. \end{aligned} \tag{77}$$

This can be compensated by introducing the following additional variation of ψ_3:

$$\delta'\Psi_3 = -\lambda\frac{1}{2}(\Psi_1\epsilon_1 + \Psi_2\epsilon_2) \,. \tag{78}$$

To summarize, in this model, the procedure of adding terms to the action and variations closes at the step $S = S_0 + S_N$. Therefore the action

$$S = \frac{1}{2}(\phi_1 - \phi_2)^2 + \phi_3^2 - \Psi_3(\Psi_1 - \Psi_2) - \lambda(\phi_1 - \phi_2)\frac{1}{2}(\Psi_1 + \Psi_2) \,, \tag{79}$$

is invariant under the following fermionic transformation:

$$\delta\phi_1 = \epsilon_1\Psi_1\,, \quad \delta\phi_2 = \epsilon_2\Psi_2\,, \quad \delta\phi_3 = \epsilon_3\Psi_3\,, \tag{80}$$

$$\delta\Psi_1 = -\phi_3\epsilon_3\,, \quad \delta\Psi_2 = +\phi_3\epsilon_3\,,$$

$$\delta\Psi_3 = (\phi_1-\phi_2)\frac{1}{2}(\epsilon_1+\epsilon_2) - \lambda\frac{1}{2}(\Psi_1\epsilon_1+\Psi_2\epsilon_2)\,,$$

$$\delta\lambda = \epsilon_1 - \epsilon_2\,. \tag{81}$$

Note that S is invariant under the exchange $\phi_1 \leftrightarrow \phi_2$ and $\Psi_1 \leftrightarrow -\Psi_2$. Note also that the 'superpartner' of the fermionic variable is missing in this model associated with P_2.

A slight modification on the action is incidentally possible and is shown as

$$S = \frac{1}{2}(\phi_1-\phi_2+\mu\phi_3)^2+\phi_3^2-\Psi_3(\Psi_1-\Psi_2)-\lambda(\phi_1-\phi_2+\mu\phi_3)\frac{1}{2}(\Psi_1+\Psi_2)\,. \tag{82}$$

This action is invariant under the following transformation:

$$\delta\phi_1 = \epsilon_1\Psi_1\,, \quad \delta\phi_2 = \epsilon_2\Psi_2\,, \quad \delta\phi_3 = \epsilon_3\Psi_3\,, \tag{83}$$

$$\delta\Psi_1 = -\left[\phi_3 + \frac{\mu}{2}\left\{\phi_1-\phi_2+\mu\phi_3-\lambda\frac{1}{2}(\Psi_1+\Psi_2)\right\}\right]\epsilon_3\,,$$

$$\delta\Psi_2 = +\left[\phi_3 + \frac{\mu}{2}\left\{\phi_1-\phi_2+\mu\phi_3-\lambda\frac{1}{2}(\Psi_1+\Psi_2)\right\}\right]\epsilon_3\,,$$

$$\delta\Psi_3 = (\phi_1-\phi_2+\mu\phi_3)\frac{1}{2}(\epsilon_1+\epsilon_2) - \lambda\frac{1}{2}(\Psi_1\epsilon_1+\Psi_2\epsilon_2)\,,$$

$$\delta\lambda = \epsilon_1 - \epsilon_2\,. \tag{84}$$

9 Concluding remarks

We have considered various analogue models for supersymmetric theory. Though they are nothing but toy models, they are in a class of models which have not ever been focused on. In higher-derivative analogue models, the simple criterion for symmetry breaking, i.e., when the partition function vanishes, is yet not clear. As far as computation of multiple integral is concerned, numerical simulation as lattice field theory may be a key tool to find the sharp criterion. The confirmation of the Ward identity in models is also expected through computational experiments.

As a simple extension of our models, incorporation of gauge symmetry can be considered, as in the original dimensional deconstruction. The non-linear sigma model on a graph is another interesting model to study. The analogue models for Lee-Wick theory with gauge fields and higher-order Galileon theory are expected to be explored. The quantum mechanical models provides a playground for defining higher-derivative field theories in the model building.

The analysis in the last section implies the difficulty in constructing 'local' models, if models have some continuum limit. Nevertheless, models on a finite graph seem to be interesting, because the structure of the models can be

interpreted as one of a mass matrix in a sense of dimensionally-deconstructed field theory. Thus, the quantum mechanical models would also be an arena of researching higher-dimensional theory.

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